Lecture 2:

Recall: Median graphs do not contain triangles, complete bipartite graphs starting from $K_{2,3}$, nor odd-lengthed cycles C_{2n+1} ($n \in \mathbb{N}^*$).

Some fundamental examples of median graphs:

→ trees

→ hypercubes

Some fundamental properties for median graphs:

I The Helly property

II The triangle condition

III The quadrangle condition

Proposition: (The Helly property)

Let X be a median graph and let $Y_1, \dots, Y_n$ ($n \in \mathbb{N}^*$) be a finite collection of convex subgraphs of X such that for all $i, j \in [\![1, n]\!]$, $Y_i \cap Y_j \neq \emptyset$.

Therefore, $\bigcap_{i=1}^{n} Y_i \neq \emptyset$.

Pf.:

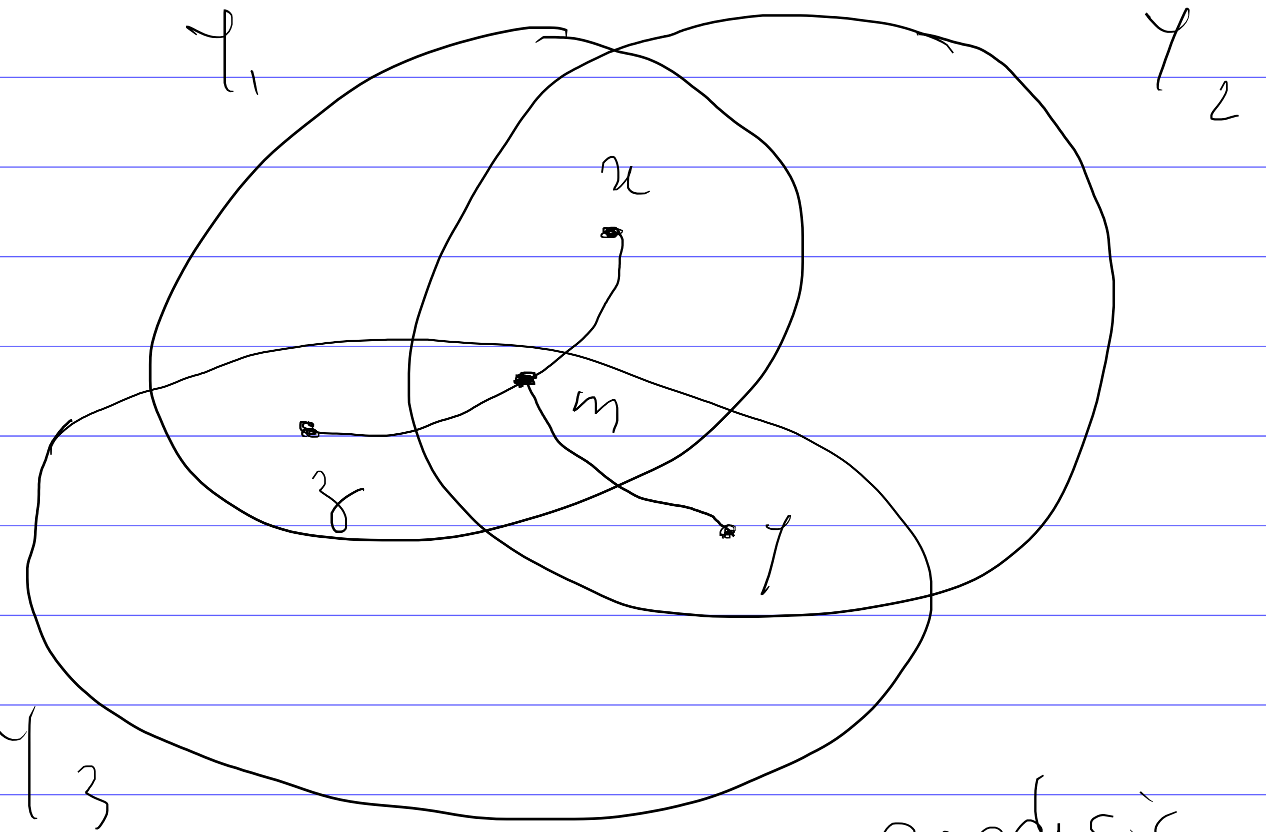
By induction on n.

$n = 3$:

Let $Y_1, Y_2, Y_3 \underset{\text{subgraphs}}{\overset{\text{convex}}{\subseteq}} X$ s.t. $Y_1 \cap Y_2 \neq \emptyset$, $Y_2 \cap Y_3 \neq \emptyset$ and $Y_1 \cap Y_3 \neq \emptyset$.

So let $x \in Y_1 \cap Y_2$, $y \in Y_2 \cap Y_3$ and $z \in Y_1 \cap Y_3$

Let $m := \mu(x, y, z)$.



geodesic convex

Because $m \in [x, y]$ and $x, y \in Y_2$, so $m \in Y_2$.
Likewise, since $m \in [x, z]$ and $m \in [y, z]$, and Y_3 and Y_1 are convex, hence $m \in Y_1$ and $m \in Y_3$.
Therefore $m \in Y_1 \cap Y_2 \cap Y_3$, so $\bigcap_{i=1}^{3} Y_i \neq \emptyset$.
Let $n \in \mathbb{N}_{\geq 3}$ be s.t. for all family $(Y_i)_{i=1}^{n}$ of convex subgraphs of X s.t. $Y_i \cap Y_j \neq \emptyset$ for all $i, j \in [\![1, n]\!]$, the intersection $\bigcap_{i=1}^{n} Y_i \neq \emptyset$.
Let now $(Y_i)_{i=1}^{n+1}$ be an $(n+1)$-family of convex subgraphs of X s.t. for all $i, j \in [\![1, n+1]\!]$, $Y_i \cap Y_j \neq \emptyset$.

Define, for all $i \in [\![1, n]\!]$, $Z_i := Y_i \cap Y_{n+1}$.
Since the intersection of convex subgraphs is a convex subgraph, so Z_i is convex for all $i \in \mathbb{N}_n$.
We also have that $Z_i \neq \emptyset$, $\forall i \in \mathbb{N}_n$.

For all $i,j \in \mathbb{N}_n$, $Z_i \cap Z_j = Y_i \cap Y_j \cap Y_{n+1}$.

By the base case, we deduce that $Z_i \cap Z_j \neq \emptyset$, $\forall i,j \in \mathbb{N}_n$.

by the induction hypothesis

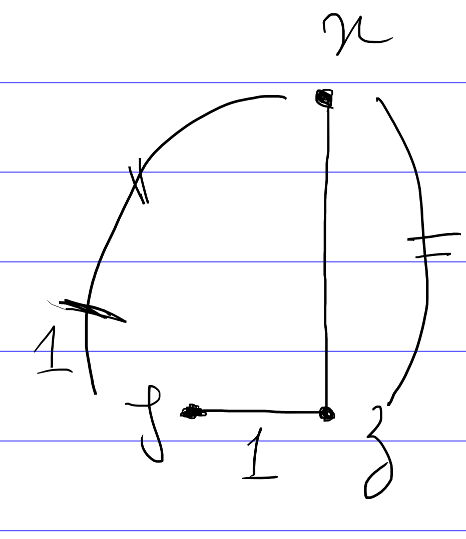
So $(Z_i)_{i=1}^{n}$ is a Helly n-family, thus, $\bigcap_{i=1}^{n} Z_i \neq \emptyset$.

But $\bigcap_{i=1}^{n} Z_i = \bigcap_{i=1}^{n+1} Y_i$, so $\bigcap_{i=1}^{n+1} Y_i \neq \emptyset$. $\square$

Proposition: (The triangle condition)

Let X be a median graph and let $x, y, z \in X$ s.t. $y \overset{1}{-} z$ (i.e., $d(y,z) = 1$).
Then, x cannot be equidistant to y and z, moreover, $d(x,z) = d(x,y) \pm 1$.

Pf.:



Let $m := \mu(x,y,z)$.

So $m \in [y,z]$, so $d(y,z) = d(y,m) + d(m,z)$

geodesic

$$1 = 1 + 0$$
$$\text{or } 0 + 1$$

WLOG, assume $d(y,m)=1$ & $d(m,z)=0$

$$\Downarrow$$

$$m \equiv z.$$

So,
$$\begin{aligned} d(x,y) &= d(x,m) + d(m,y) \\ &= d(x,z) + d(z,y) \\ &= d(x,z) + 1. \end{aligned}$$

$\square$

Property: (The quadrangle condition)

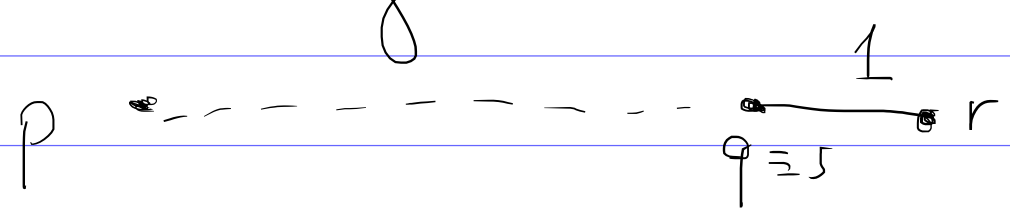
Let X be a median graph and let $p,q,r,s \in X$ in a way that r is a common neighbour of q and s (i.e., $d(q,r)=d(s,r)=1$) and $\underbrace{d(p,q)=d(p,s)}_{p \text{ is equidistant of } q \text{ and } s.}=d(p,r)-1.$

Therefore there exist a point $w \in X$ that is a common neighbour of q and s, and that verifies $d(p,w)=d(p,r)-2$.

Pf.:

Two cases.

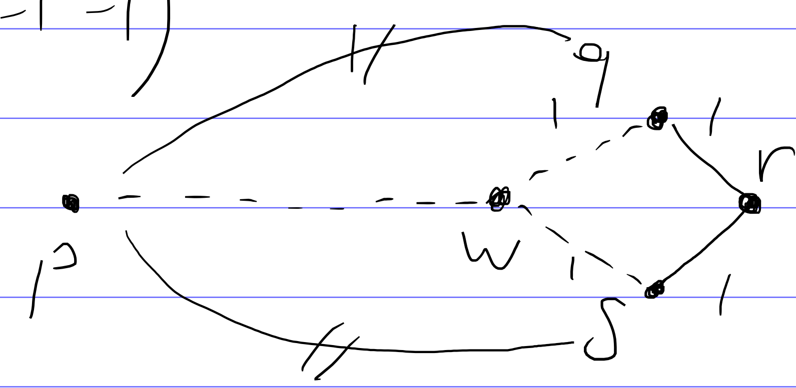
Case 1: $q \equiv s$:



In this case, it suffices to take w to be the neighbour of $q=s$ on the geodesic $\gamma=[p,q]$.

We then indeed have $d(p,w) = d(p,r) - 2$.

$\left(= d(p,q) - 1 = d(p,r) - 1 - 1\right)$



Case 2: $q \neq s$:

In that case, let $w := \mu(p,q,s)$.

Since $w \in [q,s]$ geodesic, and $d(q,s) = 2$, so

$$\begin{aligned} d(q,s) &= d(q,w) + d(w,s) \\ 2 &= \left.\begin{array}{l} 0 + 2 \\ 2 + 0 \end{array}\right\} \text{contradict the fact that } d(p,q) = d(p,s) \\ & \ 1 + 1 \end{aligned}$$

So $d(q,w) = d(s,w) = 1$, hence, w is a common neighbour of q and s.

Now,
$$\begin{aligned} d(p,r) &= d(p,q) + 1 \\ &= d(p,w) + 1 + 1 \end{aligned}$$
$$\Longrightarrow d(p,w) = d(p,r) - 2.$$

$\square$

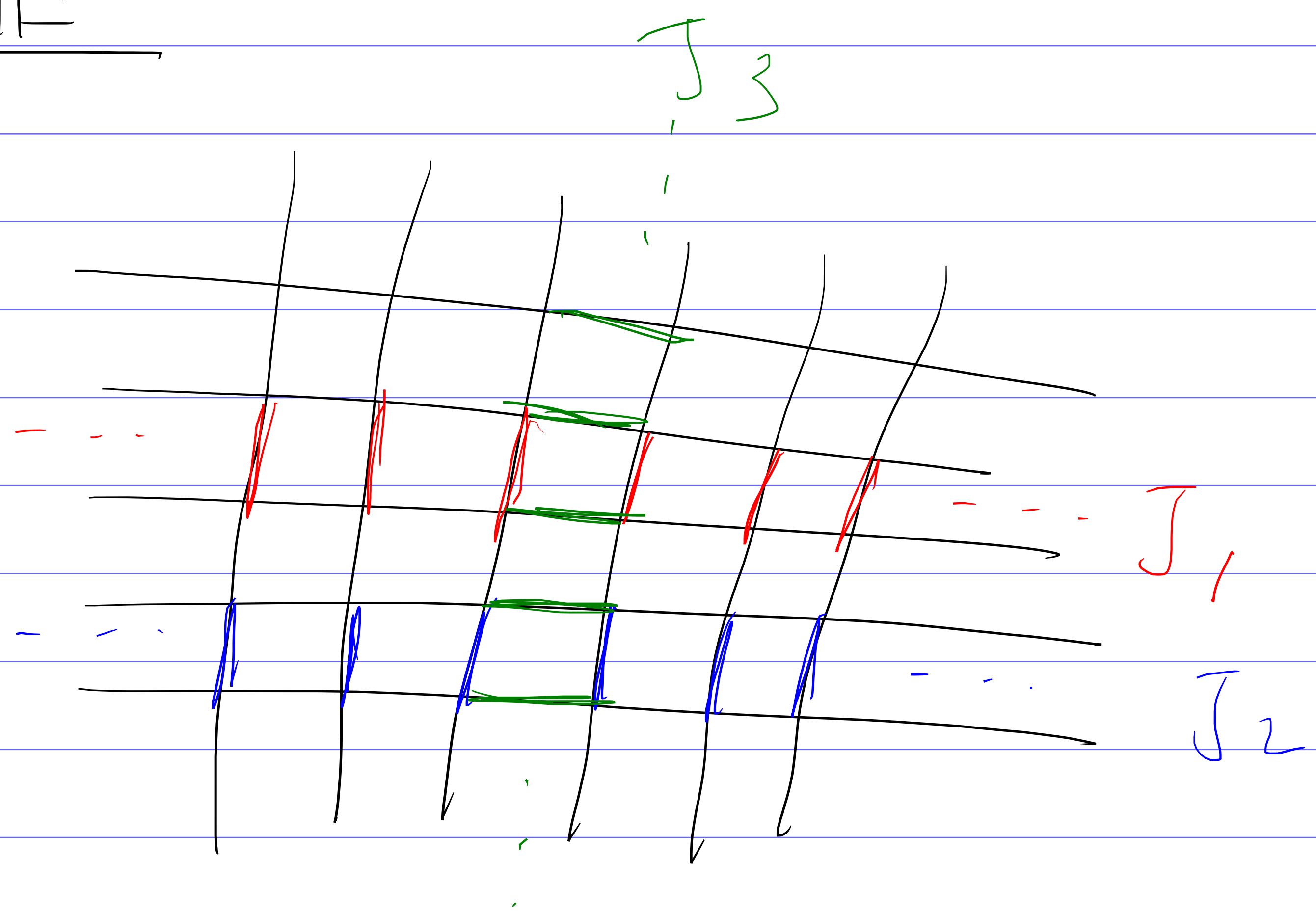
Hyperplanes and halfspaces!

Let X be a median graph.

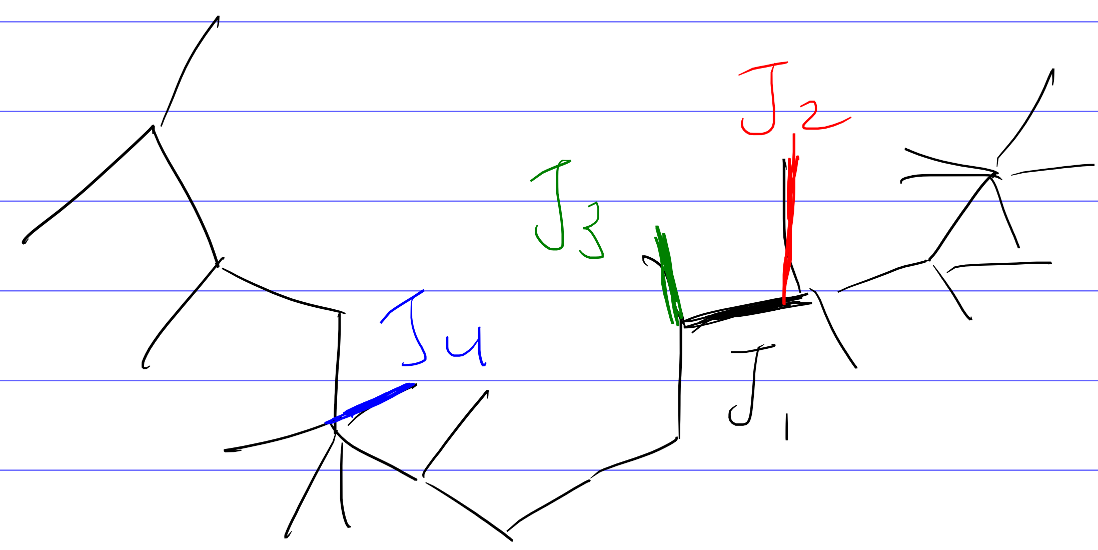
A hyperplane J in X (or of X) is an equivalence class of edges with respect to the reflexive-transitive closure of the symmetric relation that identifies two edges in X whenever they are opposite sides of a 4-cycle.

Intuitively and concretely:

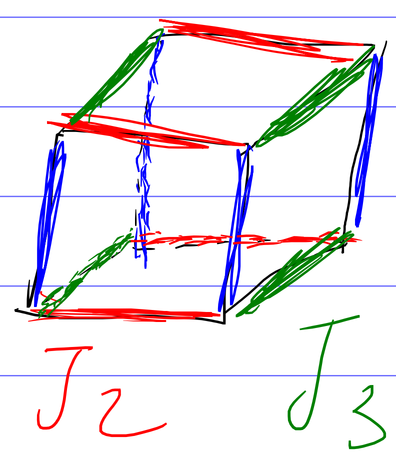
→ in $\mathbb{E}^2$



→ in a tree:



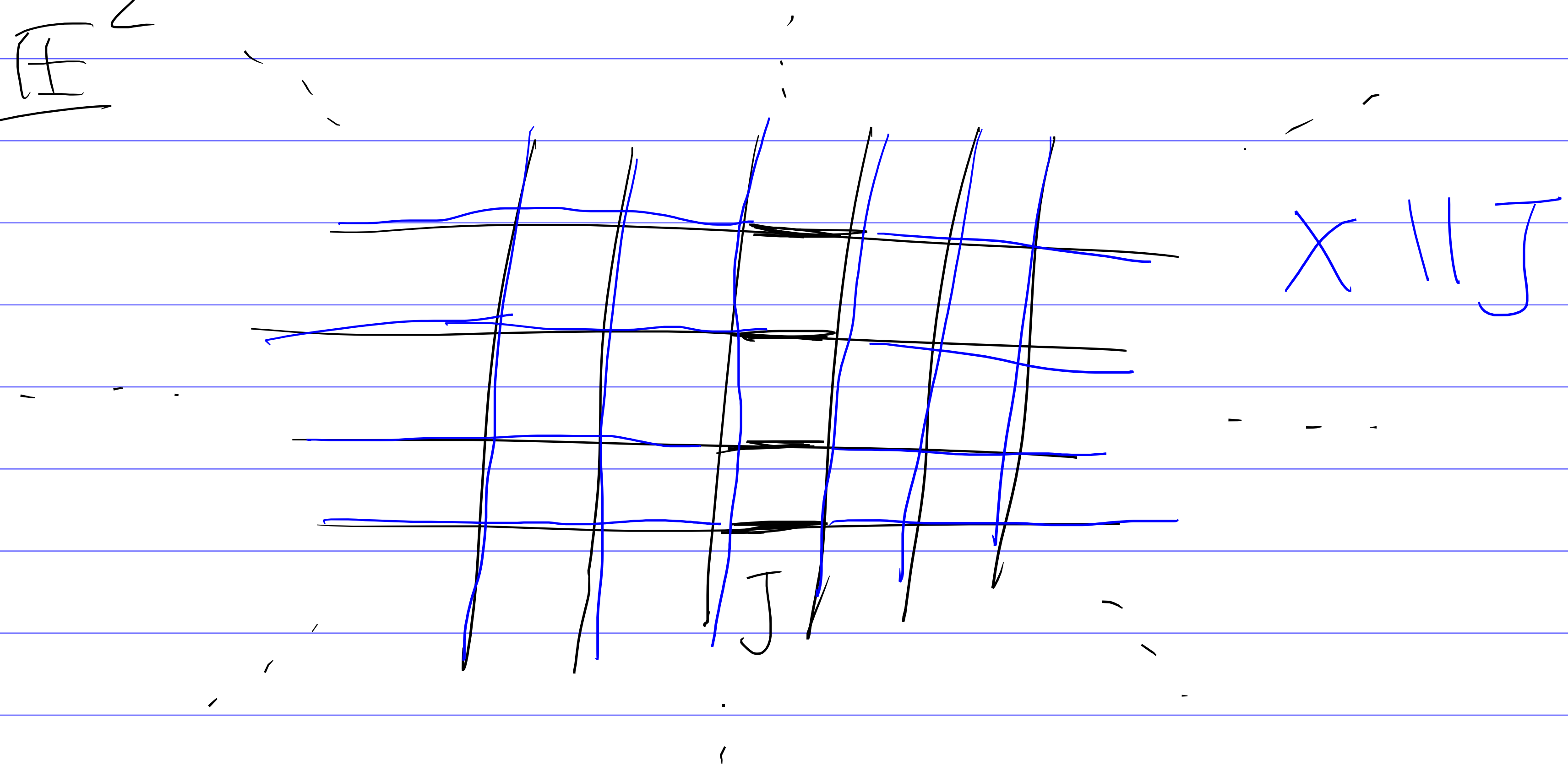
→ in a (hyper)cube:



Halfspaces:

For any hyperplane J of a median graph X, the halfspaces of J are the connected components of $X \backslash\backslash J$, where $X \backslash\backslash J$ is the graph obtained by removing the edges defining J from X.

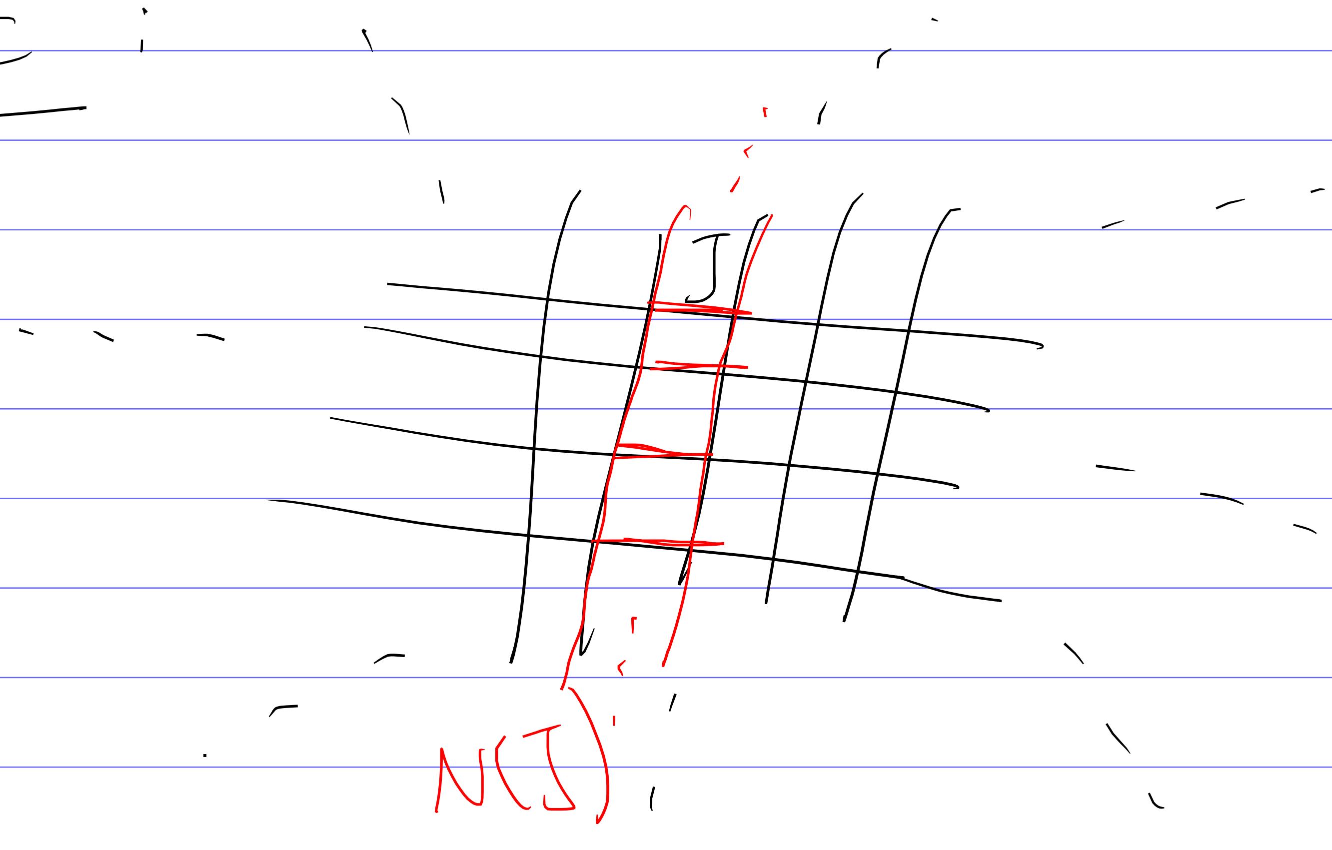
in $\mathbb{H}^2$



Neighborhoods (or carriers)

The neighborhood $N(J)$ of a hyperplane J in a median graph X is the subgraph of X constructed on all the vertices defining all the edges defining J.

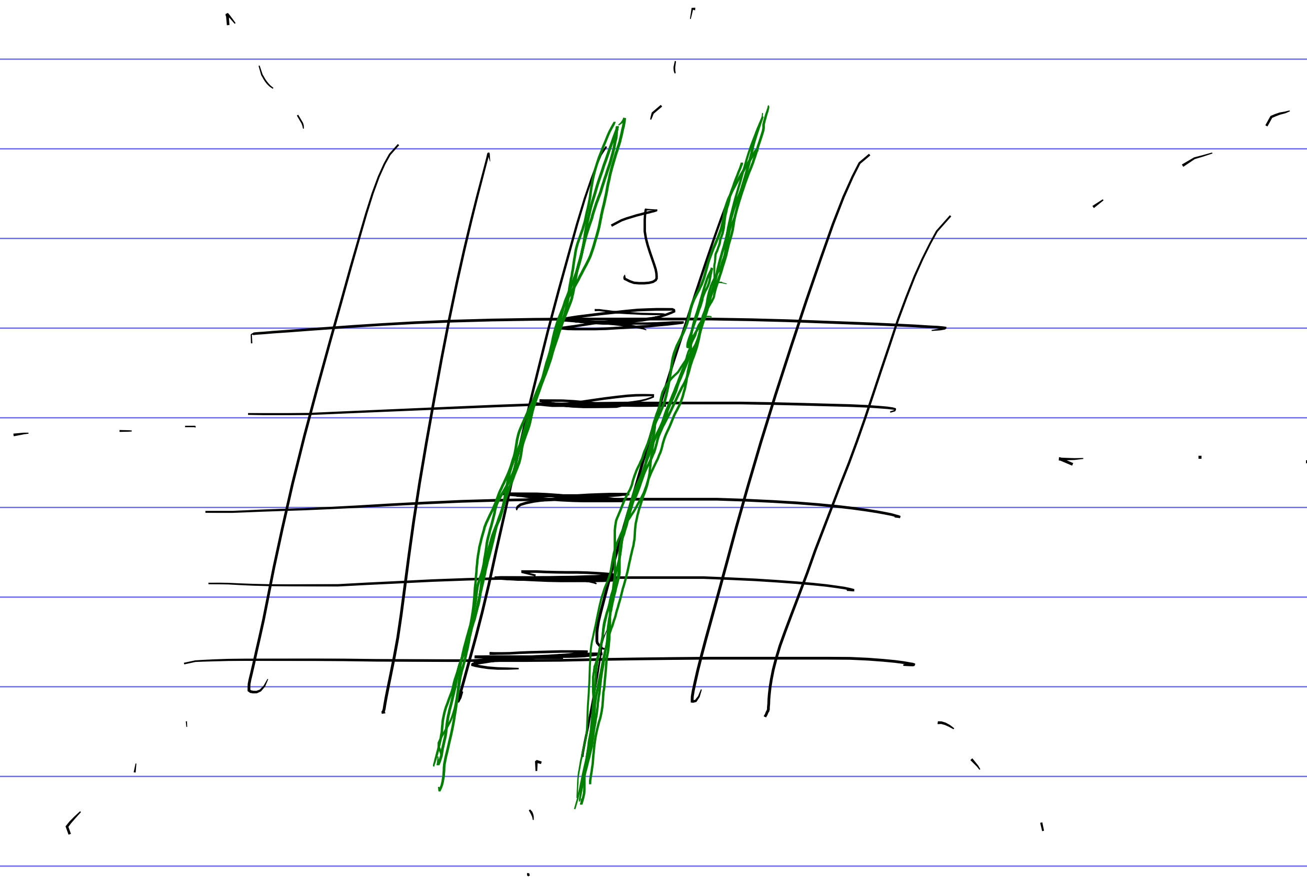
in $\mathbb{H}^2$:



Fibres:

The fibres corresponding to a hyperplane J of a median graph X are the connected components of the subgraph $N(J) \cap (X \| J)$ of X.

in $\mathbb{H}^2$:



Theorem: (big & important)

(A case where the well known fact "The geometry of a median graph reduces to the combinatorics of its hyperplanes" holds.)

Let X be a median graph.
The following propositions hold:

(i) Every hyperplane of X delimits exactly two halfspaces in X.

(ii) Carriers, halfspaces and fibres are all always convex.

(iii) Every geodesic in X crosses exactly every hyperplane in X at most once.

(iv) The distance between any two points in X is exactly equal to the number of hyperplanes of X separating these two points.